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Transport processes in dilute gases over the whole range of Knudsen numbers. Part 2. Ultrasonic sound waves

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In part 1 (Woods 1979) generalized constitutive relations for the fluid stress **p** and the heat flux **q** were established by mean-free-path arguments. These relations are expected to hold over a wide range of Knudsen numbers K, but in the earlier paper, which presented the general theory, they were successfully tested only to $O(K^2)$, the order to which the Burnett constitutive equations are valid. It remains to verify the general theory at much higher K values, and to this end we have applied it to the propagation of forced sound waves in a rarefied monatomic gas.

Theories for such waves (based on the linearized Boltzmann equation) are available, and there are also experimental results for the speed and attenuation of waves up to K values of about 10. Our theory is in good agreement with experiment in the range $0 < K < 1$. For $K > 1$ we obtain close agreement with experimental values of the wave speed, but as wave damping in this range is largely due to the unavoidable proximity of the wave transmitter – an effect not included in our calculations – we have found smaller values of the attenuation than obtaining in the experiments.

1. Introduction

In a previous paper (Woods 1979; hereinafter referred to as I) expressions were obtained for the pressure tensor **p** and heat flux vector **q** that should be accurate regardless of the value of the Knudsen number, K. The values of **p** and **q** at a point $(\mathbf{r}, t)$ in the fluid were expressed in terms of integrals over contributions coming from each element of the fluid, permitting macroscopic scale lengths (L) to be smaller than typical mean-free-paths (m.f.p.) λ. Fluid dynamics is normally confined to

$$K \equiv \lambda/L \ll 1,$$

but this constraint can be removed by generalizing the constitutive relations for **p** and **q** as just described. This approach has at least some advantages over Boltzmann's nonlinear integro-differential equation for the velocity distribution function f; for example it permits a fluid description without the need to develop a series solution, $f = f_0(1 + K\phi_1 + K^2\phi_2 + \ldots)$ where f_0 is the equilibrium distribution. Difficulties with this series and related points were discussed in I.

Burnett's constitutive equations for **p** and **q** are correct to $O(K^2)$, although of little practical value because they are subject to much the same constraint ($K \ll 1$) as are

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the Navier–Stokes equation – just slightly larger values of K are possible. In I it was shown that the integrals given there for $\mathbf{p}$ and $\mathbf{q}$ (and which will be used in this paper), when developed to $O(K^2)$, gave formulae agreeably close to Burnett's. Only one coefficient of the eleven $O(K^2)$ terms had an error exceeding 0·8 % and that was a nonlinear term which will play no role in the linear wave theory of this paper. This comparison with Burnett's work was a first test of our more general formulae.

For a second and more demanding test we propose here to study the propagation of waves of length λ_w such that $K = \lambda/\lambda_w$ varies from zero to 10 or more. Let ν be the wave frequency, τ_1 the momentum transfer collision-interval, i.e. $\tau_1 = \mu/p$, where μ is the viscosity and p the pressure, then in place of K we may adopt

$$\beta = \nu\tau_1 = \nu\mu/p. \tag{1.1}$$

Of course the viscosity mentioned here is that for normal 'fluid' conditions, namely when $\beta \ll 1$, but this imposes no restriction on the range of β in (1.1), since τ_1 is well-defined for all K. As explained in I the average m.f.p. corresponding to τ_1 is $\lambda_1 = C_1\tau_1$, where C_1 is the r.m.s. speed $(3RT)^{\frac{1}{2}}$.

Let us suppose that the waves are generated by the uniform oscillation of a plane boundary a distance l away from the point of interest in the fluid medium, then we may take ν to be real and the wavenumber $\mathrm{k} = \mathrm{k}_{\mathscr{R}} + i\mathrm{k}_{\mathscr{I}}$ to be complex. A second Knudsen number of importance is

$$\gamma = \frac{\lambda_1}{l} = \frac{C_1\beta}{\nu l} = \left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{\beta}{s} \quad \left(s \equiv \frac{\nu l}{(2RT)^{\frac{1}{2}}}\right), \tag{1.2}$$

s being an alternative parameter sometimes employed. Let a_0 be the normal sound speed, i.e. the speed at $\beta \ll 1, \gamma \ll 1$, then we define α by

$$\alpha = \alpha(\beta,\gamma) = \frac{a_0\mathrm{k}}{\nu}; \quad \alpha_{\mathscr{R}} = \frac{a_0\mathrm{k}_{\mathscr{R}}}{\nu}, \quad \alpha_{\mathscr{I}} = \frac{a_0\mathrm{k}_{\mathscr{I}}}{\nu} \quad (a_0 \equiv (\tfrac{5}{3}RT)^{\frac{1}{2}}). \tag{1.3}$$

This function defines the propagation of the ultrasonic sound waves generated by a plane wall.

In this paper we shall be content to obtain $\alpha(\beta, 0)$, namely the value of α a considerable distance from the oscillation wall or transmitter, a choice that enables us to evade the difficult problem of assigning boundary conditions on the transmitter. Some remarks on boundary conditions appear in I. The general problem will be worthy of attention at a later stage if our present theory proves successful over a β-range containing $\beta = 1$ but much less than $\beta = s$ so as to keep γ small.

Experimental results (Greenspan 1956; Meyer & Sessler 1957) are available for $0{\cdot}05 < \beta < 100$ and values of s between 5 and 11. Conflicting theories based on the linearized Boltzmann equation have been developed. Cercignani (1975) gives a general survey of this work, but does not resolve the main point at issue, namely the relative importance of the variables β and γ for the available experimental results. Sirovich & Thurber (1965*b*) argue in their defensive letter in response to severe criticisms by Maidanik & Fox (1965), that the sound characteristics depend only on β. In support of this, they show that even for γ as large as 10, 25 % of the molecules leaving the transmitter will experience a collision before moving a distance l normal to it. Thus the effective γ is reduced, suggesting that it may be sufficient to have $\beta < s$ for our

theory to be valid. However we find this no justification for following Sirovich & Thurber and completely ignoring the constraint on γ. Indeed we find their response to Maidanik & Fox's observations unconvincing, despite the good agreement they find with experiment. It is clear from Buckner & Ferziger's (1966) paper that γ is an important variable when $\beta > 1$. We shall return to this matter in § 4, when discussing the results from our theory.

A final remark is that for ease in computation, we shall develop our theory for the special case of Maxwell's molecules. That this introduces little error is clear from Sirovich & Thurber's (1965*a*) study, which compares the extreme cases of rigid sphere and Maxwell potential models. While they find that the rigid sphere model does give slightly better agreement with experiment at high frequencies, the small difference in results between taking 8 moments and 11 moments for a given model is found to be comparable to the difference obtained between the models.

2. Linearized form of the constitutive relations in non-local fluid dynamics

We shall start by summarizing the principal results for the pressure tensor and heat flux vector given in part 1. As shown in figure 1, a typical molecule S experiences a collision at $P'(\mathbf{r}', t')$ and then moves a distance R through the fluid to pass through the 'target' at $Q(\mathbf{r}, t)$, the point at which $\mathbf{p}$ and $\mathbf{q}$ are to be evaluated. The thermal speed of S when it leaves P' we write as $c'\hat{\mathbf{R}}$, where $\hat{\mathbf{R}}$ is unit vector and the speed c' is assumed to have a Maxwellian distribution. Dashes indicate values at P'. The thermal speed c and temperature T at any point are thus related by

$$c = \omega C, \quad C \equiv (2kT/m)^{\frac{1}{2}}, \tag{2.1}$$

where ω is the random variable,

$$f(\omega)\, d\omega = \frac{4}{\pi^{\frac{1}{2}}}\, \omega^2 e^{-\omega^2} d\omega \quad (0 \leqslant \omega < \infty). \tag{2.2}$$

Anisotropy in the molecular speed of S is introduced via the gradient

$$\mathscr{F}_* \equiv -\frac{1}{\rho}\, \nabla . \mathbf{p} + \omega \delta_1 \frac{1}{\rho}\, \nabla(\mathbf{p} : \hat{\mathbf{R}}\hat{\mathbf{R}} - p) \quad (\delta_1 \equiv (\tfrac{2}{3})^{\frac{1}{2}}), \tag{2.3}$$

where ρ is the fluid density and $p = (k/m)\rho T$ is the pressure. The fluid stress is assumed to act impulsively on the molecule S during its collision at P', adding a velocity component $\beta_i' \tau_i' \mathscr{F}_*'$, where the subscript is used to indicate values appropriate to momentum transport ($i = 1$) or energy transport ($i = 2$), $\tau_i', i = 1, 2$, are the mean-free-times for momentum and energy transport at P' and β_i' is a small nonlinear correction to τ_i' that we can ignore here. Let $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and $\mathbf{v}' = \mathbf{v}'(\mathbf{r}', t')$ be the fluid velocities at Q and P', and let $\mathbf{F}$ be the body force acting on the fluid, then the velocity with which S approaches Q is

$$\mathbf{u}_i = \omega C' \hat{\mathbf{R}} + \tau_i' \mathscr{F}_*' + \mathbf{v}' - \mathbf{v} + \int_{t'}^{t} \mathbf{F}\, dt, \tag{2.4}$$

the integral being along S's trajectory. For Maxwellian molecules τ_1 and τ_2 are independent of ω.

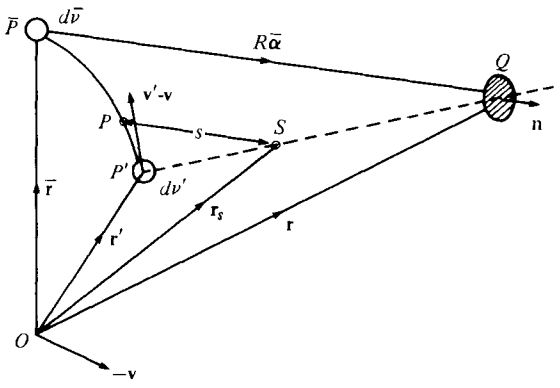

FIGURE 1. Particle trajectories: P, fluid particle molecular source; Q, molecular target; S, typical molecule.

The mean-free-paths for momentum and energy transport at $Q(\mathbf{r}, t)$ are written $\lambda_i, i = 1, 2$, and for the particular case of Maxwellian molecules are given by

$$\lambda_i = c\tau_i = \omega C\tau_i = \omega\delta_i C_i\tau_i = \omega\delta_i\lambda_i \quad (\lambda_i \equiv C_i\tau_i = \text{const.}), \tag{2.5}\dagger$$

where

$$C_1 \equiv \left(\frac{3kT}{m}\right)^{\frac{1}{2}}, \quad C_2 \equiv (\tfrac{5}{3})^{\frac{1}{2}} C_1 \quad (\delta_1 = (\tfrac{2}{3})^{\frac{1}{2}}, \quad \delta_2 = (\tfrac{4}{15})^{\frac{1}{2}})$$

are the two transport speeds appropriate to the classical theory of linear constitutive relations. At a point a distance s from P' along S's trajectory the mean-free-paths are written λ_{si}. Thus at $s = 0$, $\lambda_{si} = \lambda_i'$, and at $s = R$, $\lambda_{si} = \lambda_i$. The probability that S has a collision at P' and (later) passes through Q is+

$$\mathscr{P}_i = \frac{\exp\left[-\int_0^R ds/\lambda_{si}\right]}{4\pi R^2\lambda_i'}, \quad i = 1, 2, \tag{2.6}\dagger$$

where the integral is along the trajectory.

The fluid particle P' that we have called the molecular 'source' moves to $\bar{P}(\bar{\mathbf{r}}, t)$ during the time that S travels to the target Q, and since the mass $\rho' d\nu'$ of the fluid particle is unchanged,

$$\rho' d\nu' = \bar{\rho}\, d\bar{\nu} = \bar{\rho}\, d\tilde{\omega} R^2 dR \quad (R = |\mathbf{r} - \bar{\mathbf{r}}|),$$

where $\bar{\rho} = \rho(\bar{\mathbf{r}}, t)$ and $d\tilde{\omega}$ is the solid angle subtended at Q by $d\bar{\nu}$.

The pressure tensor and heat flux vector are given by integrals over the whole fluid volume V:

$$\mathbf{p} = \int_V \bar{\rho}\mathbf{u}_1\mathbf{u}_1\mathscr{P}_1 d\bar{\nu}, \tag{2.7}$$

and

$$\mathbf{q} = \frac{1}{2}\int_V \bar{\rho}(u_2^2 - \omega^2 C^2)\,\mathbf{u}_2\mathscr{P}_2 d\bar{\nu}, \tag{2.8}\dagger$$

† Equations (2.15), (2.12) and (2.26) of part 1 should have appeared as written in (2.5), (2.6) and (2.8) above. In the integrand of (2.8) we could have also written $\bar{\rho}(u_2^2 - \omega^2\delta_2 C_2^2)\,\mathbf{u}_2\mathscr{P}_2$.

where the bars over the integral signs indicate that the integrands are to be averaged over the Maxwellian (2.2).

In a steady, uniform state, $\mathbf{v} = 0$, $p = p_0$, $\rho = \rho_0$, $T = T_0$, when (2.4) and (2.6) reduce to

$$\mathbf{u}_i = \omega C_0 \hat{\mathbf{R}}; \quad \mathscr{P}_{0i} = \frac{e^{-R/\lambda i}}{4\pi R^2 \lambda_i}. \tag{2.9}$$

To obtain the linearized forms of (2.7) and (2.8) we write the variables as $p = p_0 + \tilde{p}$, $\rho = \rho_0 + \tilde{\rho}$, $\mathbf{v} = \tilde{\mathbf{v}}$, $C' = C_0 + \tilde{C}'$, etc., and ignore quadratic terms in the perturbation quantities $\tilde{p}, \tilde{\rho}, \ldots$. In the absence of a body force $\mathbf{F}$ (2.7) and (2.8) yield

$$\tilde{\mathbf{p}} = \overline{\int}_V \rho_0 \omega C_0 \{2\omega \tilde{C}' \hat{\mathbf{R}}\hat{\mathbf{R}} + \hat{\mathbf{R}}(\tilde{\mathbf{v}}' - \tilde{\mathbf{v}} + \tau_1 \mathscr{F}'_*) + (\tilde{\mathbf{v}}' - \tilde{\mathbf{v}} + \tau_1 \mathscr{F}'_*)\hat{\mathbf{R}} + \frac{\tilde{\tilde{\rho}}}{\rho_0} \omega C_0 \hat{\mathbf{R}}\hat{\mathbf{R}}\} \mathscr{P}_{01} d\bar{v}, \tag{2.10}$$

$$\mathbf{q} = \overline{\int}_V \rho_0 \omega^2 C_0^2 \{\omega \hat{\mathbf{R}}(\tilde{C}' - \tilde{C}) + \hat{\mathbf{R}}\hat{\mathbf{R}} . (\tilde{\mathbf{v}}' - \tilde{\mathbf{v}} + \tau_2 \mathscr{F}'_*)\} \mathscr{P}_{02} d\bar{v}. \tag{2.11}$$

A term in (2.10) containing $\tilde{\mathscr{P}}_1$ vanishes because

$$\int_V \hat{\mathbf{R}}\hat{\mathbf{R}} \mathscr{P}_1 d\bar{v} = \int_V \hat{\mathbf{R}}\hat{\mathbf{R}} (\mathscr{P}_{01} + \tilde{\mathscr{P}}_1) d\bar{v} = -\frac{1}{4\pi} \int_{\tilde{\omega}} \hat{\mathbf{R}}\hat{\mathbf{R}} \, d\tilde{\omega} \int_0^\infty d[e^{-\int \cdots}] = \tfrac{1}{3}\mathbf{I}.$$

By (2.1) the perturbation in the thermal speed C is given by

$$\frac{k}{m} \tilde{T} = C_0 \tilde{C}. \tag{2.12}$$

Also note from (2.3) that

$$\tilde{\mathscr{F}}_* = -\frac{1}{\rho_0} \nabla . [\tilde{\mathbf{p}} + \omega \delta_1 (\tilde{p} - \tilde{\mathbf{p}} : \hat{\mathbf{R}}\hat{\mathbf{R}}) \mathbf{I}], \tag{2.13}$$

where

$$\tilde{p} = \frac{k}{m} \rho_0 \tilde{T} + \frac{k}{m} \tilde{\rho} T_0. \tag{2.14}$$

To complete the equations we need the perturbation forms of the basic conservation laws of fluid mechanics, namely,

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla . \tilde{\mathbf{v}} = 0, \quad \rho_0 \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \nabla . \tilde{\mathbf{p}} = 0, \tag{2.15}$$

and

$$\frac{3}{2} \frac{k}{m} \rho_0 \frac{\partial \tilde{T}}{\partial t} + p_0 \nabla . \tilde{\mathbf{v}} + \nabla . \tilde{\mathbf{q}} = 0, \tag{2.16}$$

plus relations between the co-ordinates $(\mathbf{r}', t')$, $(\mathbf{r}, t)$ and $(\bar{\mathbf{r}}, t)$. In the uniform steady state the relations are (see figure 1)

$$\mathbf{r}' = \bar{\mathbf{r}} - R\hat{\mathbf{R}}, \quad t - t' = R/\omega C_0. \tag{2.17}$$

In the oscillatory perturbations that we shall impose on this state, the source P' at $(\mathbf{r}', t')$ will make small oscillations about its steady state location that will make small

changes to (2.17) which can be ignored in a linearized theory. For example the difference between $\tilde{C}(\mathbf{r}', t')$ and $\tilde{C}(\mathbf{r} - R\hat{\mathbf{R}}, R/\omega C_0)$ will be second order in amplitude and therefore negligible.

3. The dispersion relation for sound waves

We shall assume the existence of a plane wave with real frequency ν and complex propagation vector $\mathbf{k}$, and express all perturbation quantities in terms of

$$\xi \equiv e^{i(\mathbf{k}.\mathbf{r}-\nu t)}, \quad \chi \equiv R(\mathbf{k}.\hat{\mathbf{R}} - \nu/\omega C_0). \tag{3.1}$$

Thus by (2.17) $\rho(\bar{\mathbf{r}}, t) = \tilde{\bar{\rho}}_* e^{i(\mathbf{k}.\bar{\mathbf{r}}-\nu t)} = \rho_* \xi e^{-i\mathbf{k}.\mathbf{R}R}$, and

$$\tilde{\mathbf{v}}'(\mathbf{r}', t') = \mathbf{v}_* e^{i(\mathbf{k}.\mathbf{r}'-\nu t')} = \mathbf{v}_* \xi e^{-i\chi} = \tilde{\mathbf{v}}(\mathbf{r}, t) e^{-i\chi},$$

where $\rho_*, \mathbf{v}_*$ are constants. Similarly by (2.12)

$$C_0 \tilde{C}' = \frac{k}{m} T_* \xi e^{-i\chi}, \quad C_0 \tilde{C} = \frac{k}{m} T_* \xi.$$

Also $\tilde{\mathbf{p}} = \xi \mathbf{p}_*$, $\tilde{\mathbf{q}} = \xi \mathbf{q}_*$ and by (2.13) and (2.14)

$$\tilde{\mathscr{F}}'_* = -\frac{i}{\rho_0} \{\mathbf{k}.\mathbf{p}_* + \omega \delta_1 \mathbf{k}(p_* - \mathbf{p}_* : \hat{\mathbf{R}}\hat{\mathbf{R}})\} \xi e^{-i\chi},$$

where

$$p_* = \frac{k}{m} (\rho_* T_0 + \rho_* T_*). \tag{3.2}$$

We shall suppose that the wave is propagating along the OZ axis and that the angle between unit vector $\hat{\mathbf{k}}$ parallel to $\mathbf{k}$ and the vector $\hat{\mathbf{R}}$ is θ, as shown in figure 2. Substituting the wave forms into (2.15) and (2.16) and eliminating $\mathbf{k}.\mathbf{v}_*$, we find the relations

$$\left.\begin{aligned} \mathrm{k}^2 \hat{\mathbf{k}}\hat{\mathbf{k}} : \mathbf{p}_* &= \nu^2 \rho_*, \\ \text{and} \qquad \mathrm{k}\hat{\mathbf{k}}.\mathbf{q}_* &= -\frac{p_0}{\rho_0} \nu \rho_* + \frac{3}{2} \frac{k}{m} \rho_0 \nu T_*. \end{aligned}\right\} \tag{3.3}$$

Expressions for $\hat{\mathbf{k}}\hat{\mathbf{k}} : \mathbf{p}_*$ and $\hat{\mathbf{k}}.\mathbf{q}_*$ can be deduced from (2.10) and (2.11), which yield

$$\begin{aligned} &\mathbf{p}_* + 2i\tau_1 C_0 (\overline{\omega \mathbf{J}_s} \mathbf{k}.\mathbf{p}_*)^s - 2i\tau_1 C_0 \delta_1 (\overline{\omega^2 \mathbf{G}_1} : \mathbf{p}_* \mathbf{k})^s \\ &\quad = 2\rho_0 \frac{k}{m} T_* \overline{\omega^2 \mathbf{K}_1} + 2\rho_0 C_0 (\overline{\omega \mathbf{J}_1} \mathbf{v}_*)^s - 2i\tau_1 C_0 \delta_1 p_* (\overline{\omega^2 \mathbf{J}_1} \mathbf{k})^s + C_0^2 \rho_* \overline{\omega^2 \mathbf{L}_1}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \mathbf{q}_* = \rho_0 C_0 \frac{k}{m} T_* \overline{\omega^3 \mathbf{J}_2} &+ \rho_0 C_0^2 \mathbf{v}_* . (\overline{\omega^2 \mathbf{K}_2} - \tfrac{1}{2}\mathbf{I}) \\ &- i\tau_2 C_0^2 \mathbf{k}.\mathbf{p}_* \overline{\omega^2 \mathbf{K}_2} - i\tau_2 C_0^2 \delta_1 p_* \mathbf{k}.\overline{\omega^3 \mathbf{K}_2} + i\tau_2 C_0^2 \delta_1 \mathbf{k}\mathbf{p}_* : \overline{\omega^3 \mathbf{M}_2}, \end{aligned} \tag{3.5}$$

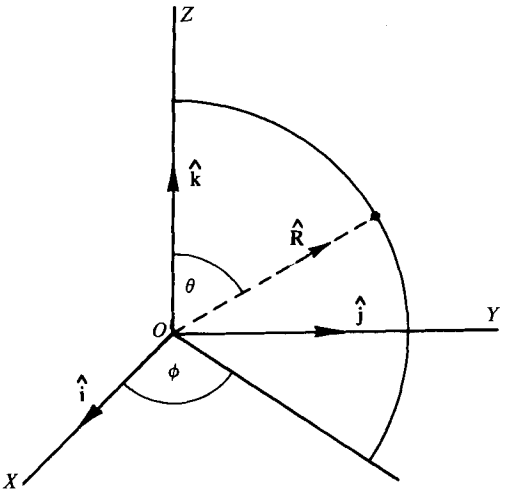

FIGURE 2. Spherical co-ordinates. $\hat{\mathbf{R}} = \hat{\mathbf{i}}\sin\theta\cos\phi + \hat{\mathbf{j}}\sin\theta\sin\phi + \hat{\mathbf{k}}\cos\theta$.

where the superscript s denotes 'symmetrical part of',

$$\left.\begin{aligned} \mathbf{J}_i &\equiv \int_V \hat{\mathbf{R}}(e^{-i\chi}-1)\,\mathscr{P}_{0i}\,d\bar{v}, \quad \mathbf{G}_i \equiv \int_V \hat{\mathbf{R}}\hat{\mathbf{R}}\hat{\mathbf{R}}\,e^{-i\chi}\mathscr{P}_{0i}\,d\bar{v}, \\ \mathbf{K}_i &\equiv \int_V \hat{\mathbf{R}}\hat{\mathbf{R}}\,e^{-i\chi}\mathscr{P}_{0i}\,d\bar{v}, \quad \mathbf{L}_i \equiv \int_V \hat{\mathbf{R}}\hat{\mathbf{R}}\,e^{-i\mathbf{k}.\hat{\mathbf{R}}R}\mathscr{P}_{0i}\,d\bar{v}, \\ \mathbf{M}_i &\equiv \int_V \hat{\mathbf{R}}\hat{\mathbf{R}}\hat{\mathbf{R}}\hat{\mathbf{R}}\,e^{-i\chi}\mathscr{P}_{0i}\,d\bar{v}, \end{aligned}\right\} \tag{3.6}$$

and the bar denotes averages as defined by

$$\bar{A} = \frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty A(\omega)\,\omega^2\,e^{-\omega^2}\,d\omega. \tag{3.7}$$

The integrals are easily evaluated. We shall take the case of Maxwellian molecules, for which $\tau_2 = \frac{3}{2}\tau_1$. Writing $x = R/\lambda_i$ and noting from (1.1), (1.3), (2.5) and (3.1) that

$$\chi = x\beta\frac{\tau_i}{\tau_1}(\omega\delta\alpha\cos\theta - 1) \quad \left(\frac{\tau_2}{\tau_1} = \tfrac{3}{2}, \quad \delta \equiv (\tfrac{6}{5})^{\frac{1}{2}}\right) \tag{3.8}$$

we have by (2.9) that

$$e^{-i\chi}\mathscr{P}_{0i}\,d\bar{v} = \exp\left[-x(1-\cos\theta/b_i)/h_i\right]\sin\theta\,d\theta\,\frac{d\theta}{4\pi}\,dx,$$

and

$$\exp\left[-i\mathbf{k}.\hat{\mathbf{R}}R\right]\mathscr{P}_{01}\,d\bar{v} = \exp\left[-x(1-\cos\theta/b_0)\right]\sin\theta\,d\theta\,\frac{d\phi}{4\pi}\,dx,$$

where

$$b_0 \equiv \frac{i}{\omega\delta\alpha\beta}, \quad b_i \equiv \frac{i+\dfrac{\tau_i}{\tau_1}\beta}{\dfrac{\tau_i}{\tau_1}\omega\delta\alpha\beta}, \quad h_i \equiv \frac{1}{1-i\dfrac{\tau_i}{\tau_1}\beta} \quad (i=1,2). \tag{3.9}$$

Thus

$$\mathbf{K}_i = \frac{1}{4\pi}\int_0^\infty \int_0^{2\pi} \int_0^\pi (\hat{\mathbf{i}}\sin\theta\cos\phi + \hat{\mathbf{j}}\sin\theta\sin\phi + \hat{\mathbf{k}}\cos\theta)(\ldots) \times \exp[-x(1-\cos\theta/b_i)/h_i]\sin\theta\, d\theta\, \frac{d\phi}{4\pi}\, dx.$$

Let

$$\phi_i \equiv 1 + \tfrac{1}{2}b_i \ln\left(\frac{1-b_i}{1+b_i}\right) \quad (i = 0, 1, 2) \tag{3.10}$$

then

$$\mathbf{K}_i = \tfrac{1}{2}h_i\{(b_i^2-1)\,\phi_i + 1\}(\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}}) - h_i b_i^2 \phi_i \hat{\mathbf{k}}\hat{\mathbf{k}}. \tag{3.11}$$

Similarly

$$\mathbf{L}_1 = \tfrac{1}{2}\{(b_0^2-1)\,\phi_0 + 1\}(\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}}) - b_0^2\phi_0\,\hat{\mathbf{k}}\hat{\mathbf{k}}, \tag{3.12}$$

$$\mathbf{J}_i = -h_i b_i \phi_i \hat{\mathbf{k}}, \tag{3.13}$$

$$\hat{\mathbf{k}}.\mathbf{G}_1 = \tfrac{1}{2}h_1 b_1\{\tfrac{1}{3} + (b_1^2-1)\,\phi_1\}(\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}}) - h_1 b_1(\tfrac{1}{3} + b_1^2\phi_1)\,\hat{\mathbf{k}}\hat{\mathbf{k}}, \tag{3.14}$$

and

$$\hat{\mathbf{k}}\hat{\mathbf{k}}:\mathbf{M}_2 = \tfrac{1}{2}h_2 b_2^2\{\tfrac{1}{3} + (b_2^2-1)\,\phi_2\}(\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}}) - h_2 b_2^2(\tfrac{1}{3} + b_2^2\phi_2)\,\hat{\mathbf{k}}\hat{\mathbf{k}}. \tag{3.15}$$

(As we shall see shortly, only the components of $\mathbf{G}_1$ and $\mathbf{M}_2$ given here will be required.)

To form a convenient set of scalar equations we take the products $\hat{\mathbf{k}}\hat{\mathbf{k}}$: and $(\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}})$: of (3.4) and the product $\hat{\mathbf{k}}.$ of (3.5), then eliminate p_*, $\hat{\mathbf{k}}\hat{\mathbf{k}}:\mathbf{p}_*$ and $\hat{\mathbf{k}}.\mathbf{q}_*$ from the resulting equations by (3.2) and (3.3). The result can be expressed as

$$\begin{bmatrix} \mathscr{A} & \mathscr{B} & \mathscr{C} \\ \mathscr{D} & -1 & \mathscr{F} \\ \mathscr{G} & \mathscr{H} & \mathscr{I} \end{bmatrix} \begin{bmatrix} \rho_* \\ (\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}}):\mathbf{p}_*/C_0^2 \\ (\rho_0/C_0^2)\,kT_*/m \end{bmatrix} = [0], \tag{3.16}$$

where $\mathscr{A}, \mathscr{B}, \ldots, \mathscr{I}$ are the non-dimensional coefficients:

$$\mathscr{A} = 1 + \frac{2(i+\beta)}{\beta}\overline{\phi_1} - \frac{\overline{\phi_0}}{\beta^2} + \delta_1\left\{\delta^2\alpha^2\overline{\omega\phi_1} - \frac{4}{3\sqrt{\pi}} - \frac{2(i+\beta)^2}{\delta^2\alpha^2\beta^2}\left(\frac{\overline{\phi_1}}{\omega}\right)\right\},$$

$$\mathscr{B} = \delta_2\alpha^2\delta^2\left\{\frac{2}{3\sqrt{\pi}} + \frac{(i+\beta)^2}{\delta^2\alpha^2\beta^2}\left(\frac{\overline{\phi_1}}{\omega}\right) - \overline{\omega\phi_1}\right\},$$

$$\mathscr{C} = \frac{2i(i+\beta)}{\beta^2}\overline{\phi_1} + 2\delta_1\delta^2\alpha^2\overline{\omega\phi_1}, \quad \mathscr{D} = \tfrac{3}{4} - \tfrac{1}{2}\overline{\omega^2\phi_0} - \frac{\overline{\phi_0}}{\alpha^2\delta^2\beta^2},$$

$$\mathscr{F} = \frac{i(i+\beta)}{\delta^2\alpha^2\beta^2}\overline{\phi_1} + \frac{i}{i+\beta}(\tfrac{3}{2} - \overline{\omega^2\phi_1}),$$

$$\mathscr{G} = \frac{4}{9}\frac{(i+\tfrac{3}{2}\beta)^2}{\delta^2\alpha^2\beta^2}\overline{\phi_2} + \tfrac{1}{3}\delta_1\frac{i+\tfrac{3}{2}\beta}{\beta}\overline{\omega\phi_2} - \tfrac{2}{3}\delta_1\frac{i+\tfrac{3}{2}\beta}{\delta^2\alpha^2\beta^2}\left\{\frac{2}{3\sqrt{\pi}} + \frac{4}{9}\frac{(i+\tfrac{3}{2}\beta)^2}{\delta^2\alpha^2\beta^2}\left(\frac{\overline{\phi_2}}{\omega}\right)\right\},$$

$$\mathscr{H} = \tfrac{1}{3}\delta_1\frac{i+\tfrac{3}{2}\beta}{\beta}\left\{\frac{2}{3\sqrt{\pi}} + \frac{4}{9}\frac{(i+\tfrac{3}{2}\beta)^2}{\delta^2\alpha^2\beta^2}\left(\frac{\overline{\phi_2}}{\omega}\right) - \overline{\omega\phi_2}\right\},$$

$$\mathscr{I} = \tfrac{3}{2} + \tfrac{2}{3}i\frac{\overline{\omega^2\phi_2}}{\beta} + \tfrac{2}{3}\delta_1\frac{i+\tfrac{3}{2}\beta}{\beta}\overline{\omega\phi_2}.$$

From (3.16) it follows that these coefficients are subject to the constraint

$$\mathscr{A}\mathscr{I} - \mathscr{C}\mathscr{G} + \mathscr{H}(\mathscr{A}\mathscr{F} - \mathscr{C}\mathscr{D}) + \mathscr{B}(\mathscr{D}\mathscr{I} - \mathscr{F}\mathscr{G}) = 0, \tag{3.17}$$

which is therefore the required relation between α and β, i.e. the dispersion relation.

When β is small (3.17) reduces to

$$1-\alpha^2+\tfrac{7}{5}i\beta\alpha^2=0,$$

which yields

$$\alpha_{\mathscr{R}}=1, \quad \alpha_{\mathscr{I}}=\tfrac{7}{10}\beta, \tag{3.18}$$

results that one can easily show are in agreement with those from the Navier–Stokes equations for ordinary sound waves. For other values of β it is necessary to solve (3.17) numerically. This has been done by Mr John Cady of the Computing Methods Unit of the NSW Institute of Technology, Sydney. In the following section we shall compare the results obtained with experimental results and with conclusions obtained from other theories.

4. Comparison with other theories and with experiment

In figures 3 and 4 are plotted $\alpha_{\mathscr{R}}$ and $\alpha_{\mathscr{I}}$ as functions of β^{-1}. The experimental results shown due to Meyer & Sessler (1957) were obtained with Ar at $s = 8{\cdot}25$, where s is the number defined in (1.2). Greenspan's observations (with He, Ne, Ar, Kr, and Xe), at least those shown in the figure, have $s > (\frac{3}{2})^{\frac{1}{2}}\beta$, since he excluded results for which $\gamma > 1$. Hence from the remarks made in the sixth paragraph of the Introduction, we should not expect our theory to hold much beyond $\beta \approx 1$. In fact for the wave speed $(a_0/\alpha_{\mathscr{R}})$ we have very good agreement up to $\beta = 100$, with only Buckner & Ferziger's (1966) theory closer to the experimental points.

The increased wave speed at a given point P for $\beta \gg 1$ is because the slower molecules arriving at P from neighbouring regions have phases varying more widely than do the faster molecules arriving at P, which means that they cannot contribute as much to the sound pressure as the faster molecules (Maidanik, Fox & Heckl 1965). Provided there are some collisions between molecules in the gas, this remark is true whether a transmitter is close to P or not. While the presence of a nearby transmitter will enhance the phase mixing at P of molecules of all speeds, its effect will be largely independent of speed because of the wide range of distances between P and points on the transmitter. We therefore do not expect the wave *speed* to depend on the closeness of the transmitter, and conclude from this and figure 3 that our theory is entirely satisfactory as far as $\alpha_{\mathscr{R}}$ is concerned.

On the other hand, at first sight, our theory does not appear to have successfully predicted wave *damping* beyond $\beta \approx 1$. Damping at $\beta \gg 1$ is largely due to phase mixing, and we have just asserted that the presence of a surface W (the transmitter) near P will increase this mixing. If this is indeed true, then the discrepancy in figure 4 between our theory and experiment is possibly explicable as being due to 'wall damping'. Damping of waves due to particles colliding with nearby walls is not a new concept. Woods (1965) in studying the propagation of density waves in the positive column, found considerable wave damping in a collisionless plasma due to ions being lost to the surrounding boundaries. Of course molecules are not lost on striking the transmitter W, but assuming that diffuse reflexion occurs, the molecules coming from somewhere near P, striking W and then returning to P, will be as mixed in phase as possible, certainly more so than molecules coming to P from the direction of W were W absent.

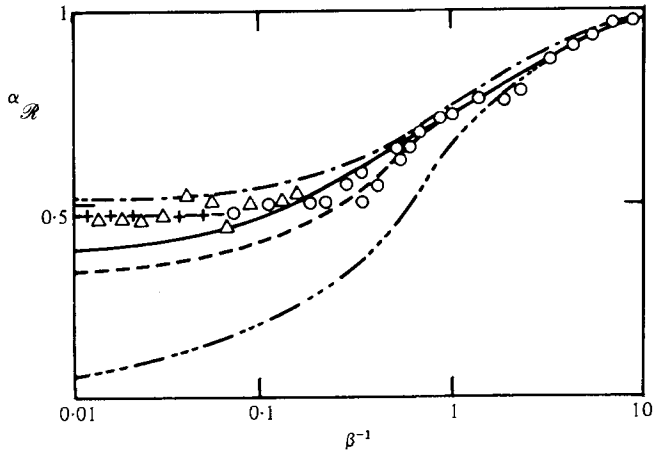

FIGURE 3. $\alpha_{\mathscr{R}}$ *vs.* β^{-1}. Theory: ——, Sirovich & Thurber (11 moments); — —, Sirovich & Thurber (8 moments): — + —, Buckner & Ferziger (5 moments); — — —, Woods & Throughton; — · — · —, Navier–Stokes. Experimental data: △, Mayer & Sessler; ○, Greenspan.

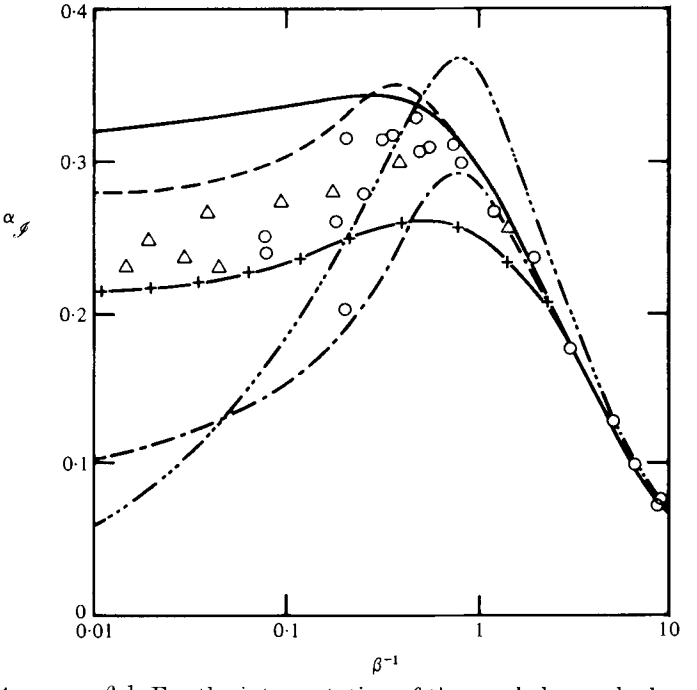

FIGURE 4. $\alpha_{\mathscr{I}}$ *vs.* β^{-1}. For the interpretation of the symbols see the legend to figure 3.

Several authors have used Boltzmann's equation to obtain asymptotic expressions for α at large values of β. The most accurate appears to be that given by Hanson & Morse (1969), who show that in the limit $\beta \to \infty$,

$$\alpha \sim (\tfrac{5}{12})^{\frac{1}{2}} 2^{-\frac{1}{8}} (\sqrt{3} - i)\, s^{-\frac{1}{3}} \quad (s = (\tfrac{3}{2})^{\frac{1}{2}} \beta/\gamma). \tag{4.1}$$

Maidanik, Fox & Heckl (1964) give a similar formula, but with a different coefficient. This expression shows that as the transmitter surface W moves away from P, the

damping due to phase mixing slowly declines, which supports our concept of 'wall damping'. But there appears to be no published analytic solution for the limit $\gamma \to 0$, with which to compare our results. Sirovich & Thurber's (1965*a*) calculations appear to be independent of the value of γ, so presumably hold for $\gamma = 0$. They claim that (4.1) is not appropriate to the experimental results, with which they get fair agreement, save that the more accurate 11 moments model is further from experiment than the 8 moments model. Buckner & Ferziger's (1966) method, which does incorporate the value of γ, appears to us to be the most reliable of those based on the linearized Boltzmann equation. It remains an open question whether our theory is in error for $\beta > 1$ or not.

One of us (L. C. W.) was working by invitation of Dr B. S. Thornton (Head of the School of Mathematical Sciences) at the N.S.W. Institute of Technology, Sydney, when the substantial part of the work described above was completed. And we are also particularly grateful to Mr John Cady of the Institute for his numerical solution of our rather complicated dispersion relation, which was by no means a simple task.

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